The Phase Transition of the Directed Polymer on Disordered Hierarchical Lattices

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The directed polymer on disordered hierarchical lattices is studied using an exact renormalization scheme. The phase transition is studied and a hyperscaling relation is derived. The fixed distributions of the renormalized energies are obtained numerically. The specific heat and associated exponents are computed from the fixed distributions.

1. INTRODUCTION

The problem of directed polymers in random media has been extensively studied in recent years.[1-10] The directed polymer provides a simple but nontrivial disordered model which is related to a variety of other problems, such as growth models, spin glasses, etc. For $1 + 1$ dimensions the system is always in the low-temperature phase and the exponents are known exactly.[4] However, for high dimensions the nature of the low-temperature phase is not yet fully understood. Derrida and Griffiths[5] studied the problem by considering directed polymers on disordered hierarchical lattices, mainly because the problem is much simpler than in the case of Euclidean lattices. They derived an exact nonlinear zero-temperature recursion relation for the random energies and got some interesting results. For dimension $1 + 1$ they found $\omega = 0.30$, which is very close the exact value $1/3$, where L^{ω} is the ground-state energy fluctuation of a polymer of length L. Other interesting properties of the problem have also been studied in recent papers.[9, 10]

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For sufficiently high dimensions, it is found that there is a phase transition between the low-temperature phase and a high-temperature phase.[6-8] Cook and Derrida[8] extended the approach used in ref. 5 to the finite-temperature case. They obtained an exact recursion relation for the partition function. By analyzing the properties of the moments of the partition function under iteration, they showed that there is a critical point separating the low- and high-temperature phases for sufficiently high dimensions. They also computed the specific heat curve, which shows a cusp at the critical temperature. However, it is not easy to calculate the specific heat exponent from their method because the partition function grows indefinitely under iteration and has no fixed point.

In this paper, we use a renormalization method in analogy with the Migdal–Kadanoff renormalization group (MKRG) for random spin systems.[11-16] The fixed point for the phase transition is found to be associated with a fixed distribution of the renormalized energies. We also derive a hyperscaling relation and use it to obtain the specific heat exponent.

2. THE MODEL AND THE RECURSION RELATION

Following ref. 8, generalized hierarchical lattices can be constructed by an iterative rule where each brach has k bonds (see Fig. 1), at each new generation the number of the bonds is multiplied by kb, and at the nth generation the hierarchical lattices consists of $(kb)^{n-1}$ bonds. Disorder is introduced by assigning a random energy ε_{ij} for each bond ij according to a given probability distribution $\rho(\varepsilon_{ij})$. On these disordered hierarchical lattices, the directed polymer starts at A and ends at B. At the nth
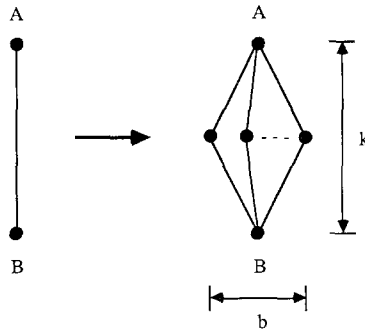

Fig. 1.   The iterative construction of the generalized hierarchical lattice.

generation the length of the polymer is $L = k^{n-1}$. The recursion relation for the partition function is

$$Z_{n+1} = Z_n^{(11)} Z_n^{(12)} \cdots Z_n^{(1k)} + \cdots + Z_n^{(b1)} Z_n^{(b2)} \cdots Z_n^{(bk)} \tag{1}$$

where Z_{n+1} is the partition function at generation $n+1$ and the $Z_n^{(ij)}$ are the kb partition functions at generation n. The average free energy per unit length of the polymer at the nth generation can be written

$$F_n(T) = -T \frac{\langle \log(Z_n) \rangle}{k^{n-1}} \tag{2}$$

At the fixed point, $F_n(T_c)$ becomes independent of n. From Eq. (2), we have $\langle \log(Z_n) \rangle \sim k^{n-1}$, which is strictly increasing with n, so Eq. (1) has no fixed point. To find the fixed point, we follow the renormalization group method for spin systems[17] and introduce a multiplicative constant to the partition function at each stage. From (1) we have

$$Q_{n+1} = \frac{Q_n^{(11)} Q_n^{(12)} \cdots Q_n^{(1k)} + \cdots + Q_n^{(b1)} Q_n^{(b2)} \cdots Q_n^{(bk)}}{c_n} \tag{3}$$

where Q_{n+1} is the partition function at generation $n+1$ in our new scheme, and $Q_n^{(ij)}$ is the kb partition functions at generation n. We have $Q_1^{(ij)} = e^{-(\varepsilon_{ij}/T)}$, where ε_{ij} represent the initial random energies. The renormalized energies at the nth generation are

$$\varepsilon_n = -T \log(Q_n) \tag{4}$$

From (3) and (4), one can easily see that the effect of c_n is only to shift all energies by an amount of $-T \log(c_n)$ at the nth generation. Since the absolute value of the energies is not important, only the variation of the energies determines the physical properties; this freedom permits us to set the mean value of the renormalized energies to zero,

$$\langle \log Q_n \rangle = 0 \tag{5}$$

Then from (3) and (5), we have

$$\log(c_n) = \langle \log(Q_n^{(11)} Q_n^{(12)} \cdots Q_n^{(1k)} + \cdots + Q_n^{(b1)} Q_n^{(b2)} \cdots Q_n^{(bk)}) \rangle \tag{6}$$

and c_n is determined uniquely. The difference between our method and that of ref. 8 is that we only keep the fluctuation of the random energies, while in ref. 8 the absolute energies are kept to all iterations. In other words, we have the relation $\varepsilon_n/T = -[\log(Z_n) - \langle \log(Z_n) \rangle]$, as we shall see in the

Appendix. At the fixed point, the distribution of ε becomes invariant under iteration.

The average of the free energy per unit length can be obtained directly from the additive constant $\log(c_n)$ in analogy with the spin systems[11,12,17]

$$F_n(T) = -T \sum_{i=1}^{n-1} \frac{\log(c_i)}{k^i} \tag{7}$$

In the Appendix, we show that (2) and (7) are equivalent to each other. One advantage of (7) is that c_n grows much more slowly than Z_n; for example, at the fixed point c_n becomes a constant. So it is more suitable to compute the free energy using (7) especially in high dimensions, where Z_n grows rapidly with n.

From Eq. (7), the hyperscaling relation can be derived in a standard way.[17] A very simple method for the derivation was given by da Cruz and Stinchcombe.[14] Let v be the relevant exponent which is associated with the fixed distribution, we have

$$\alpha = 2 - v \tag{8}$$

where α is the specific heat exponent. Note that the hyperscaling law does not explicitly depend on dimension, which is different from the spin systems.

To see the condition for the phase transition, we let the variance of the initial energy be very small and the average value be zero. Using (4), we expand (6) to second order in ε_n and take the average over disorder; we get

$$\log(c_n) = \log(b) + \frac{kb - k}{2b} \langle \varepsilon_n^2 \rangle \tag{9}$$

We substitute (9) into (4), then do the same kind of expansion, square both sides, average over disorder, and keep the lowest order; we get

$$\langle \varepsilon_{n+1}^2 \rangle = \frac{k}{b} \langle \varepsilon_n^2 \rangle \tag{10}$$

For $k/b > 1$, the variance increases under iteration, the directed polymer is always in the low-temperature phase, and no phase transition occurs. For $k/b < 1$, the variance decreases under iteration, the directed polymer is in the high-temperature phase, and as we increase the initial variance, we need to keep higher terms in (9) and the system may make a transition to the low-temperature phase; a phase transition occurs. In particular, let $k = 2$; the condition for phase transition requires $b > 2$, in agreement with ref. 8.

3. NUMERICAL RESULTS

In this paper we use a Monte Carlo method to study the phase transition. Several numerical approaches have been developed to study the critical properties of various random systems on hierarchical lattices.[12, 15] Since the fixed point is unstable under renormalization, we use a version of Newton's method to locate the fixed distribution of the renormalized energies. The distribution is represented by 150,000 numbers and is transformed according to (3) and (6). We first narrow down the initial condition onto a phase boundary; the distribution flows to the neighborhood of the fixed point. We assume that the fixed distribution is controlled by the second moment of the energies. If the root mean square σ of the random energies changes into σ' under renormalization, then each renormalized random energy is scaled by a factor $[1 - 0.2(\sigma' - \sigma)]\sigma/\sigma'$. In other words, the distribution is adjusted in the direction opposite to the change. If the assumption is correct, then the distribution should converge to the fixed distribution. The simulation is carried out for $k = 2$, $b = 3$ and $k = 2$, $b = 5$. After many iterations we find σ is 0.56 ± 0.03 for $b = 3$ and 0.78 ± 0.05 for $b = 5$. It would be interesting to check this result with some other methods, such as the histogram method.[12] From Fig. 2, one can clearly see that the fixed distribution is not symmetric, unlike the case for the spin-glass system.[13]

To calculate the exponent, we use a very similar procedure to ref. 15; we first find the fixed distribution by the above procedure, then a second copy of the ensemble of the random energies is made. We multiply each
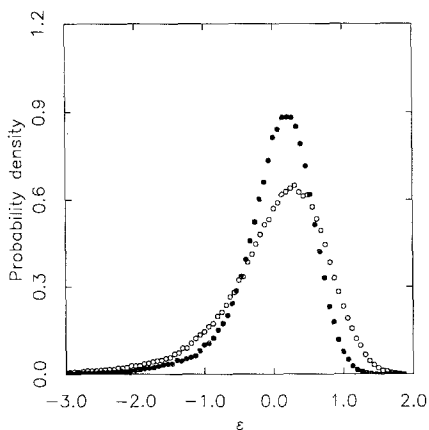

Fig. 2. The fixed distributions of the random energy for $b = 5$ (open circles) and $b = 3$ (solid circles).

random energy by a number f which is slightly larger than one; we choose $f = 1.0001$. The two copies of the distribution are then simultaneously transformed according to the above procedure, and the difference of σ between the two copies at each successive iteration is recorded, δ_n; then the exponent v is given by

$$v = \frac{\log(k)}{\log(\delta_n/\delta_{n-1})} \tag{11}$$

After many iterations, we take the average value of δ_n/δ_{n-1}. We find $v = 3.8 \pm 0.3$ for $b = 3$ and $v = 2.7 \pm 0.1$ for $b = 5$. The value of α can be obtained from the hyperscaling law (10). We get $\alpha = -1.8$ for $b = 3$ and $\alpha = -0.7$ for $b = 5$.

In ref. 8, the specific heat curves were calculated numerically; the initial energies were chosen from the Gaussian distribution, and the curve displays a cusp for $b = 5$. In this paper, we choose the fixed distribution as the initial distribution, and using our new formula (7), we compute the specific heat curves. The cusp we obtain for $b = 5$ has a much bigger amplitude than the one in ref. 8 (see Fig. 3). This is not surprising, since if we start from an arbitrary random distribution close to the phase boundary, the distribution would flow to the fixed distribution first and then drift either to the low-temperature phase or high-temperature phase. Note that after each iteration the free energy is divided by a factor of 2 [see (2) and (7)] and therefore the amplitude is reduced. We also calculated the curve for $b = 3$ (see Fig. 4); however, the peak is rounded at the critical temperature due to a large, negative α.

In the MKRG scheme, several analytical methods[15, 16] have been developed to study the critical properties of diluted spin systems with a
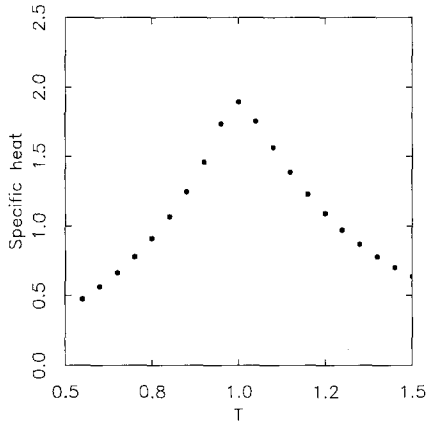

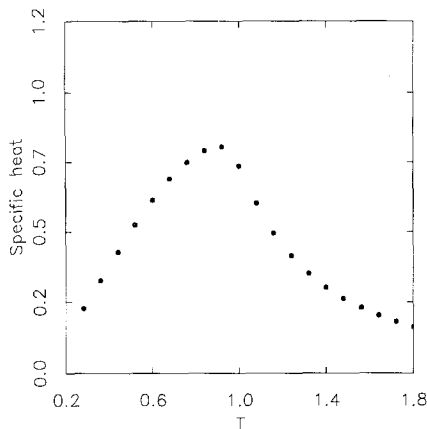Fig. 3.   The specific heat curve for $b = 5$.

Fig. 4.   The specific heat curve for $b = 3$.

positive α where the fixed point is also associated with a fixed distribution. If the fixed distribution is narrow, using a moment expansion method, one gets some rather satisfying results. However, if the distribution is wide, these methods may have some errors. In our case, we calculated the first five moment of the renormalized energies for $b = 3$, which is the smallest integer to allow the phase transition. Unfortunately, we found that these moments are the same order. This indicates that the analytic calculations may not be feasible.

4. CONCLUSION

We have seen that the phase transition is associated with a fixed distribution of the renormalized energies. The distribution either spreads under renormalization in the low-temperature phase or shrinks in the high-temperature phase. The average of the renormalized energies is always kept at zero. This is very similar to the transition between the spin-glass phase and the paramagnet phase for the Ising model, except that the fixed distribution is not symmetric. We think that the hyperscaling law derived here also holds in the Euclidean space as in the case of the spin systems. Unfortunately, the wandering exponent cannot be computed on the hierarchical lattices. The numerical method we have developed also can be applied to the spin-glass transition on hierarchical lattices.[13] A similar phase transition has been found for the self-avoiding random walk in a quenched random medium if $d > 4$.[18] However, if the disorder is correlated in a special way,[19] the phase transition can occur in three dimensions, and the specific heat displays a cusp.

APPENDIX. THE EQUIVALENCE OF (2) AND (7)

To save writing, we only discuss the case $k = 2$, $b = 2$; the generalization to other cases is straightforward. We use the induction method to prove that our approach is consistent with ref. 8. Starting from the same initial distribution, we have

$$Z_1^{(i)} = Q_1^{(i)} \tag{A1}$$

Using (A1), (2), and (3), we have the following relation:

$$
\begin{aligned}
Z_2^{(i)} &= Z_1^{(1)}Z_1^{(2)} + Z_1^{(3)}Z_1^{(4)} \\
&= (Q_1^{(i)}Q_1^{(i)} + Q_1^{(i)}Q_1^{(i)}) \\
&= c_1 Q_2^{(i)}
\end{aligned} \tag{A2}
$$

Suppose at the nth generation we have

$$Z_n^{(i)} = \exp\left[\sum_{i=1}^{n-1} \log(c_i)\, 2^{n-i-1}\right] Q_n^{(i)} \tag{A3}$$

One can easily see that when $n = 2$, (A1) is recovered from (A3). For the $(n+1)$th generation, we obtain

$$
\begin{aligned}
Z_{n+1}^{(i)} &= Z_n^{(1)}Z_n^{(2)} + Z_n^{(3)}Z_n^{(4)} \\
&= \left\{\exp\left[\sum_{i=1}^{n-1} \log(c_i)\, 2^{n-i-1}\right]\right\}^2 (Q_n^{(1)}Q_n^{(2)} + Q_n^{(3)}Q_n^{(4)}) \\
&= \exp\left[\sum_{i=1}^{n-1} \log(c_i)\, 2^{n+1-i-1}\right] c_n Q_{n+1}^{(i)} \\
&= \exp\left[\sum_{i=1}^{n+1-1} \log(c_i)\, 2^{n+1-i-1}\right] Q_{n+1}^{(i)}
\end{aligned} \tag{A4}
$$

so we have proved (A3). Now we can derive (7). From (2), (A3), and (5), we have

$$
\begin{aligned}
F_n(T) &= -T \frac{\langle \log(Z_n) \rangle}{2^{n-1}} \\
&= -T\left[\sum_{i=1}^{n-1} \frac{\log(c_i)}{2^i} + \frac{\langle \log(Q_n) \rangle}{2^{n-1}}\right] \\
&= -T \sum_{i=1}^{n-1} \frac{\log(c_i)}{2^i}
\end{aligned} \tag{A5}
$$

From (4), (A3), and (A5), the renormalized energy can be obtained by

$$\frac{\varepsilon_n}{T} = -[\log(Z_n) - \langle \log(Z_n) \rangle]$$ (A6)

For the more general case, 2 is replaced by k, while b does not get involved explicitly.

ACKNOWLEDGMENTS

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